THE KING'S SCHOOL

2008
Higher School Certificate Course
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

Blank Page

Total marks – 120
Attempt Questions 1-10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate xe^x **2**

(b) $f(x) = x^3 - 3x^2 - 6x - 6$

Find $f'(-1)$ **2**

(c) Solve $(x - \sqrt{2})^2 = 4$ **2**

(d) Find a primitive of $\sec^2 2x$ **2**

(e) Find, correct to two decimal places, $\log_{12} 2008$ **2**

(f) Factorise $a^3 + 8b^3$ **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

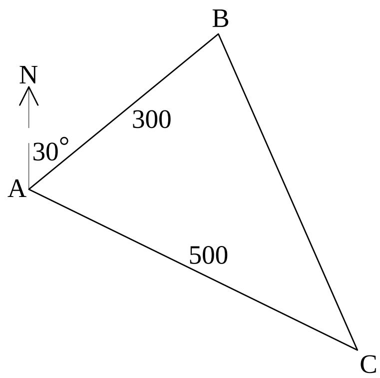
(a) Solve $\tan\theta + 1 = 0$ for $0 < \theta < 2\pi$ **2**

(b) Find

(i) $\displaystyle\int \frac{x}{x^2 + 1}\, dx$ **2**

(ii) $\displaystyle\int \frac{x^2 + 1}{x}\, dx$ **2**

(c)



A ship sails from A for 300 km on a bearing of 030˚ to B. Another ship sails from A for 500 km on a bearing of 150˚ to C.

(i) Show that $\angle BAC = 120^\circ$. **1**

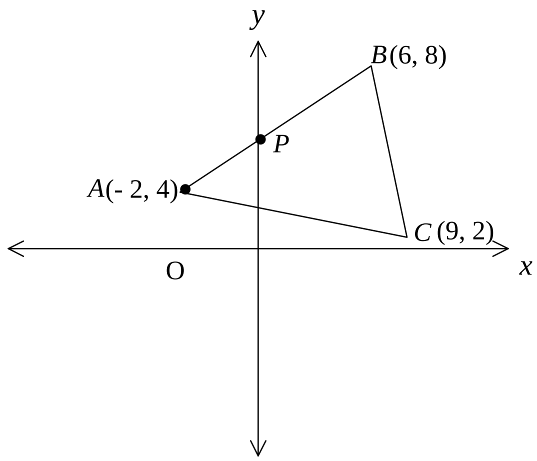
(ii) Use the cosine rule to show that BC = 700 km. **2**

(iii) Use the sine rule to find the bearing of C from B.
Give your answer correct to the nearest degree. **3**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The vertices of ΔABC are (- 2, 4), (6, 8), (9, 2), respectively. The line *AB* meets the *y* axis at *P*.

(i) Find the gradient of *AB*. **1**

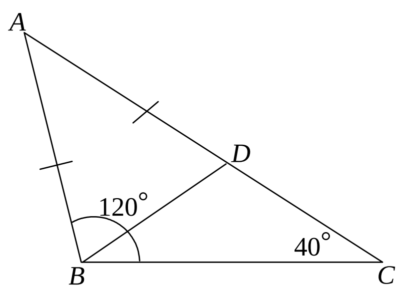
(ii) Hence, or otherwise, find the *y* coordinate of *P*. **2**

(iii) By finding the gradient of *BC*, or otherwise, show that $\angle ABC = 90^\circ$. **2**

(iv) Find the area of ΔABC. **2**

(v) Hence find the height of the triangle using BC as its base. **1**

(b)



In the diagram, $\angle BCA = 40^\circ$, $\angle ABC = 120^\circ$, $AB = AD$

(i) Prove that $\angle ADB = 80^\circ$ **2**

(ii) Deduce that $BD = CD$ **2**

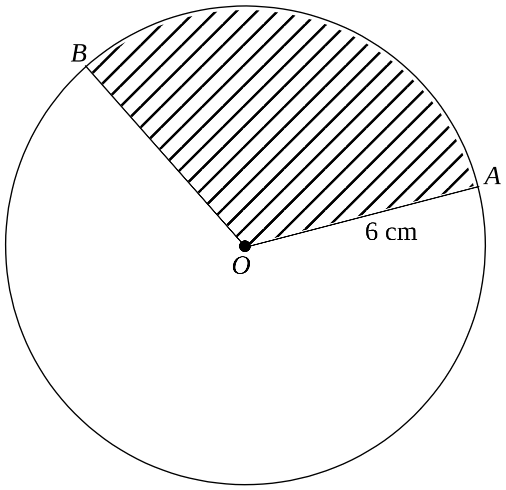
End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of $x^2 - 12x + 6 = 0$ are α, β.

Evaluate $\alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$ **3**

(b)



The diagram shows a sector of a circle, centre O, and radius 6 cm. The area of the sector is $12\pi\ \text{cm}^2$.

(i) Find $\angle AOB$ **2**

(ii) Find the length of the arc AB of the sector. **1**

(c) A particular curve $y = f(x)$ has a stationary point (0, 0). Also, $f''(x) = 2e^{2x} - 2$.

(i) Show that at (0, 0) there is a horizontal point of inflection. **2**

(ii) Show that $f'(x) = e^{2x} - 2x - 1$ **2**

(iii) Find $f(1)$ **2**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the hyperbola $y = \frac{1}{x+1}$ showing any intercepts made on the axes. **2**

(ii) The region bounded by the hyperbola $y = \frac{1}{x+1}$ and the x axis from $x = 0$ to $x = k > 0$ is revolved about the x axis.

Prove that the volume of the solid of revolution is less than π for all values of $k > 0$. **4**

(b) Let $P(x) = -x^2 + (k+2)x - 1$

(i) Sketch $y = P(x)$ for $k = -4$ **2**

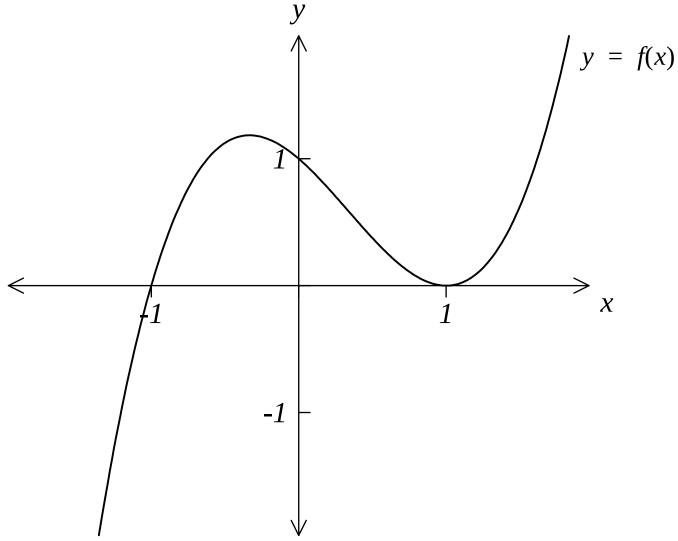
(ii) For what values of k does $P(x) = 0$ have real roots? **4**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = \sin(x^2 + 2x)$ at the point where $x = -2$. **4**

(b)



The diagram shows the sketch of $y = f(x)$.

Sketch the graph of $y = f'(x)$. **2**

(c) $P(198, 998)$ is a point on the parabola $(x+2)^2 = 4000(1008 - y)$. The directrix of the parabola is the line $y = 2008$.

Find the distance from P to the focus. **2**

(d) Simplify $\frac{a}{(a-b)(a-c)} - \frac{b}{(b-c)(a-b)} + \frac{c}{(a-c)(b-c)}$ **2**

(e) Find the range of the function $y = \ln(2 + \sin x)$. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use Simpson's Rule with three function values to give a two decimal place approximation to $\int_0^{\frac{\pi}{3}} 2\tan^3 x \, dx$. **3**

(b) (i) Show that $\frac{d}{dx}(\tan^2 x) = 2\tan x + 2\tan^3 x$ **2**

(ii) Hence, or otherwise, find $\frac{d}{dx}(\sec^2 x)$ **1**

(iii) Use (i) to show that $\int_0^{\frac{\pi}{3}} 2\tan^3 x \, dx = 3 - 2\ln 2$ **3**

(c) $1 + 2x + 4x^2 + \ldots$ is a geometric series.

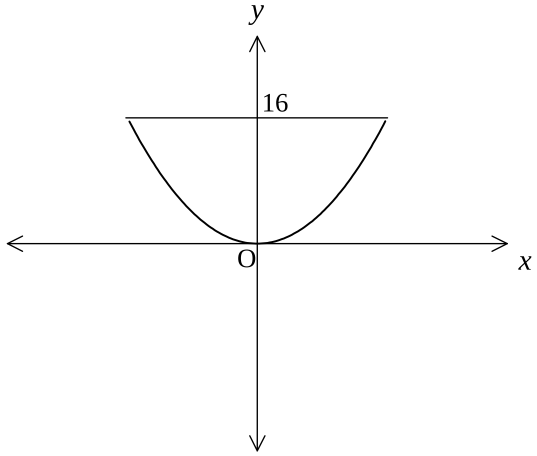
(i) For what values of x does the limiting sum exist? **1**

(ii) Is it possible for the limiting sum to be 12? Give reasons. **2**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the region enclosed between the curve $y = 16x^4$ and the line $y = 16$.

Find the area of the region. **3**

(b) (i) Albert deposited \$2 000 each year for 20 years into a fund paying 12% p.a. simple interest. Find the interest Albert made over the 20 years. **3**

(ii) Betty deposited \$2 000 each year for 20 years into a fund paying 7% p.a. compound interest. The interest was compounded annually.

Who made the better financial decision? Albert or Betty? **4**

(c) Solve $\ln x^2 - \ln x - 12 = 0$ **2**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A cylinder, open at one end, is to have a volume of 1728π cm^3.

$$[V = \pi r^2 h, \text{ Curved Surface Area } = 2\pi rh]$$

Let the radius of the cylinder be r cm and the height h cm.

(i) Show that the total surface area S is given by $S = \pi\left(r^2 + \dfrac{3456}{r}\right)$. **2**

(ii) Prove that the minimum surface area is 432π cm^2. **4**

(b) Sketch the curve $y = -\cos\left(\dfrac{x}{2}\right)$ for $-4\pi \leq x \leq 4\pi$. **2**

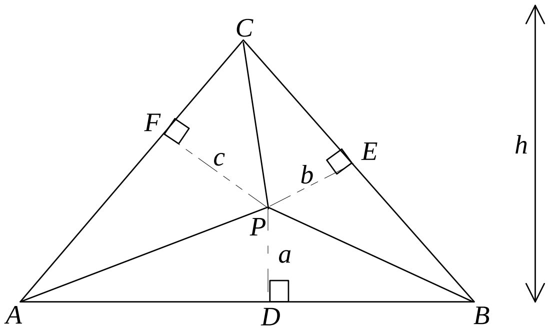
(c) (i) Solve $|x + 2| \leq 5$. **1**

(ii) Hence, or otherwise, solve $1 \leq |x + 2| \leq 5$. **3**

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i)

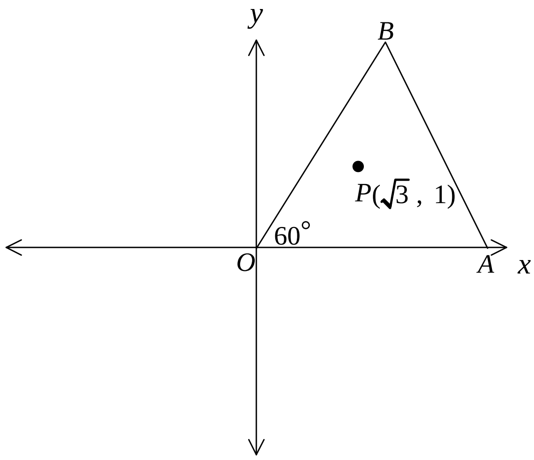


ΔABC is equilateral and P is any point interior to the triangle.

Perpendiculars PD, PE and PF are drawn to the sides AB, BC and CA, respectively.

Let $PD = a$, $PE = b$ and $PF = c$

If h is the altitude of the triangle, by considering areas of triangles, show that $h = a + b + c$. **2**



ΔOAB is equilateral. O is the origin. A is on the x axis and B has the x coordinate 2.

$P(\sqrt{3}, 1)$ is a point interior to the triangle.

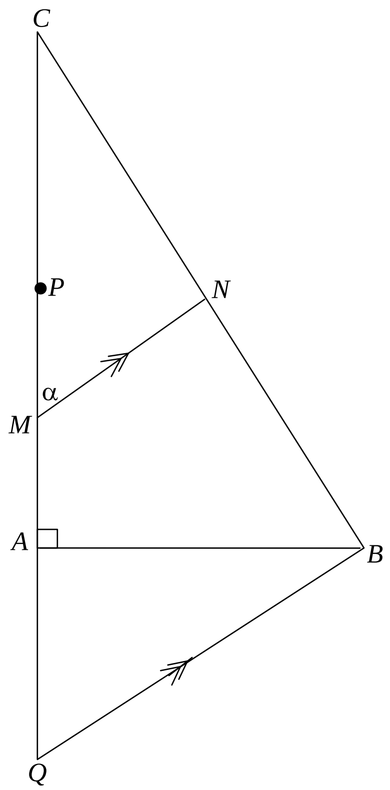
(ii) Show that the equation of OB is $y = \sqrt{3}\,x$. **2**

(iii) Find the perpendicular distance from $P(\sqrt{3}, 1)$ to the line OB. **2**

(iv) Hence, or otherwise, find the perpendicular distance from $P(\sqrt{3}, 1)$ to the line AB. **2**

Question 10 continues on next page

(b)



The diagram shows ΔABC, right-angled at A. P is the point on CA so that $CP = AB$.

M is the mid-point of AP and N is the mid-point of BC.

Q is the point on CA produced so that QB is parallel to MN.

Let $\angle CMN = \alpha$.

Prove that α is a fixed value. **4**

End of Examination

Blank Page

Blank Page

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

The King's School

2008
Higher School Certificate
Trial Examination

Mathematics

Question	Algebra and Number	Geometry	Functions	Trigonometry	Differential Calculus	Integral Calculus	Total
1	(c), (e), (f) 6				(a), (b) 4	(d) 2	12
2				(a), (c) 8		(b) 4	12
3		(b) 4	(a) 8				12
4			(a), (b) 6			(c) 6	12
5			(a)(i), (b) 8			(a)(ii) 4	12
6	(d) 2		(c), (e) 4		(a), (b) 6		12
7	(c) 3				(d)(i)(ii) 3	(a), (b)(iii) 6	12
8	(b), (c) 9					(a) 3	12
9	(c) 4			(b) 2	(a) 6		12
10	(a) 8	(b) 4					12
Total	32	8	26	10	19	25	120

TKS MATHEMATICS TRIAL 2008 SOLUTIONS

Question 1

(a) $xe^x + e^x$

(b) $f'(x) = 3x^2 - 6x - 6$

$\therefore f'(-1) = 3 + 6 - 6 = 3$

(c) $x - \sqrt{2} = \pm 2 \quad \therefore x = \sqrt{2} \pm 2$

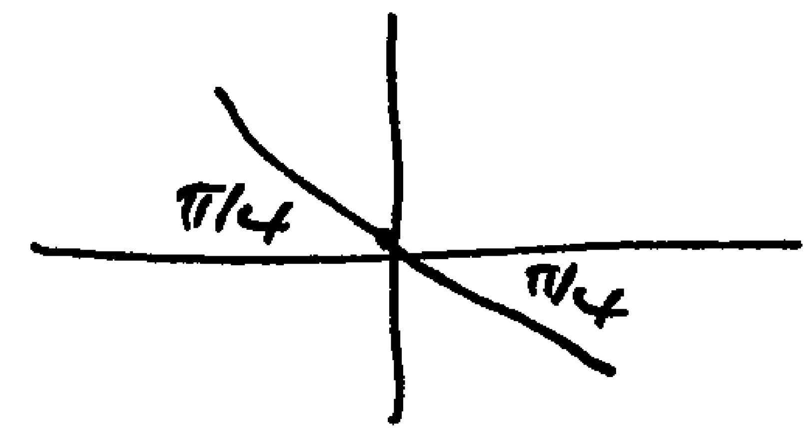
(d) $\frac{1}{2} \tan 2x$

(e) $\log_{12} 2008 = \dfrac{\ln 2008}{\ln 12} = 3 \cdot 06$, 2 d.p.

(f) $(a + 2b)(a^2 - 2ab + 4b^2)$

Question 2

(a) $\therefore \tan\theta = -1 \Rightarrow$

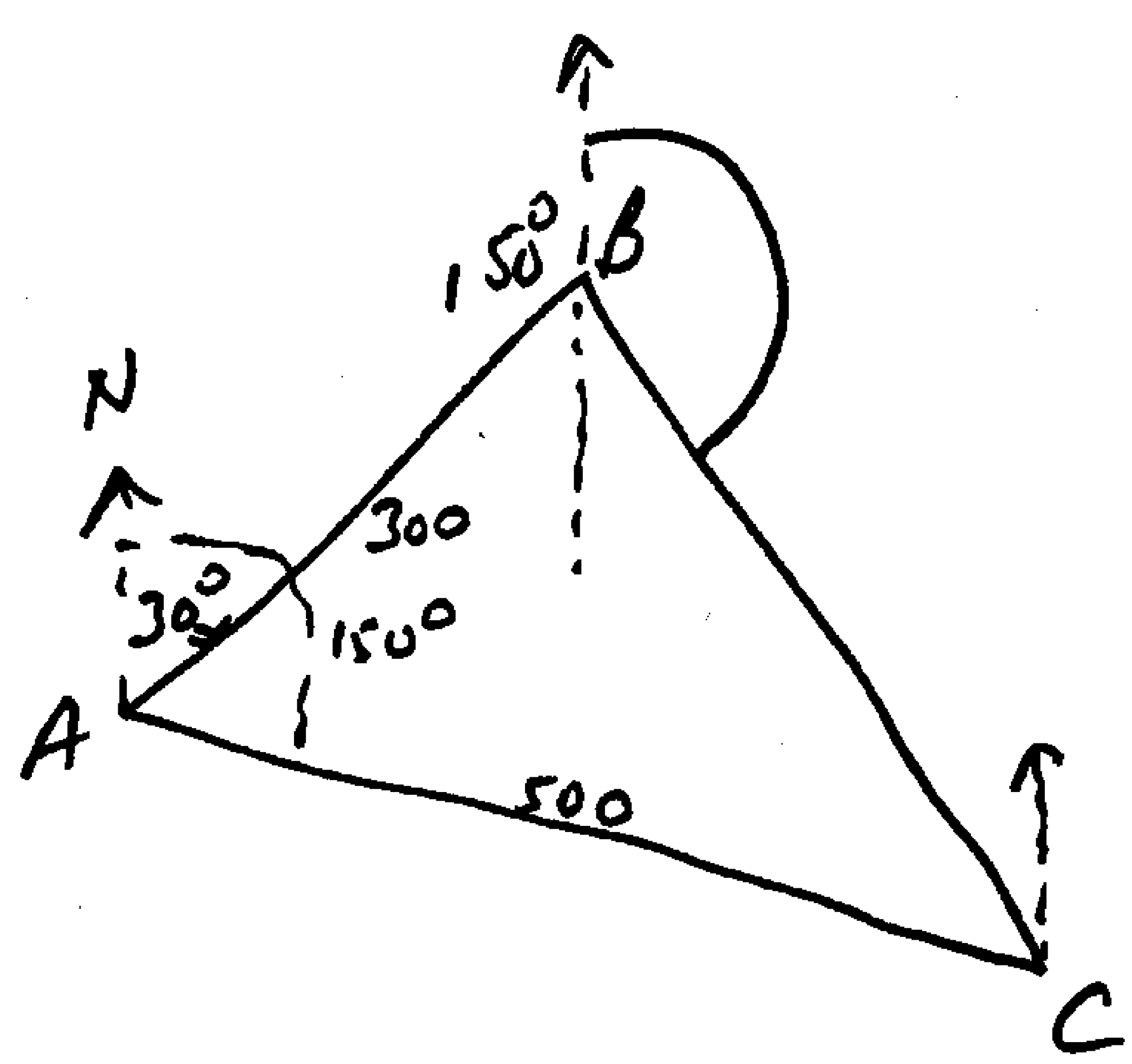


$\therefore \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

(b) (i) $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$

(ii) $\int x + \frac{1}{x}\,dx = \frac{x^2}{2} + \ln x$

(c) (i)



$\angle BAC = \angle NAC - \angle NAB = 150° - 30° = 120°$

(ii) $BC^2 = 300^2 + 500^2 - 2 \times 300 \times 500 \cos 120°$

$= 490\,000$

$\Rightarrow BC = \sqrt{490000} = 700$

(iii) $\frac{\sin B}{500} = \frac{\sin 120°}{700} \quad \therefore \sin B = \frac{500 \sin 120°}{700}$

$\hat{B} = 38°$, nearest degree

$\therefore$ bearing of C from B $= 360° - (150° + 38°)$

$= 172°$

Question 3

(a) (i) grad $AB = \dfrac{8-4}{6--2} = \dfrac{4}{8} = \dfrac{1}{2}$

(ii) take $P(0,y) \Rightarrow \dfrac{y-4}{0--2} = \dfrac{1}{2}$

$\therefore y-4 = 1, \quad y = 5$

(iii) grad $BC = \dfrac{6}{-3} = -2$

& since grad $AB \times$ grad $BC = -1$, then $AB \perp BC$

$\Rightarrow \angle ABC = 90^\circ$

(iv) $AB = \sqrt{8^2+4^2} = \sqrt{80}$

$BC = \sqrt{3^2+6^2} = \sqrt{45}$

$\therefore$ Area $\triangle ABC = \frac{1}{2} \cdot \sqrt{80} \cdot \sqrt{45} = 30$

(v) $\therefore 30 = \frac{1}{2} \cdot \sqrt{45} \cdot h \Rightarrow h = \dfrac{60}{\sqrt{45}}$ will do [or $\dfrac{20}{\sqrt{5}}$ or $4\sqrt{5}$]

(b) (i) $\angle BAC = 180^\circ - (120^\circ + 40^\circ)$, $\angle$ sum $\triangle ABC$

$= 20^\circ$

In isosceles $\triangle ABD$, $\angle B = \angle D = \frac{1}{2}(160^\circ) = 80^\circ$, $\angle$ sum $\triangle ABD$

ie $\angle ADB = 80^\circ$

(ii) $\angle DBC + \angle DCB = \angle ADB$, ext $\angle$ theorem in $\triangle BDC$

$\Rightarrow \angle DBC = 80^\circ - 40^\circ = 40^\circ$

$\therefore \triangle BDC$ is isosceles, base angles equal

$\therefore BD = CD$

Question 4

(a) $\alpha+\beta = 12, \quad \alpha\beta = 6$

$$\therefore \alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta} = \alpha+\beta+\frac{\alpha+\beta}{\alpha\beta} = 12+2 = 14$$

(b) (i) Let $\angle AOB = \theta$

Then $\frac{1}{2}\cdot 6^2\cdot\theta = 12\pi \Rightarrow \theta = \frac{12\pi}{18} = \frac{2\pi}{3}$

(ii) $AB = 6\cdot\frac{2\pi}{3}$ cm $= 4\pi$ cm

(c) (i) $f''(0) = 2-2 = 0$

$f''(-1) = 2e^{-2}-2 < 0$
$f''(1) = 2e^2-2 > 0$ $\Rightarrow$ change in concavity

$\therefore (0,0)$ is a horiz. pt. of inflection since it's a stat. pt

(ii) $f'(x) = 2\frac{e^{2x}}{2} - 2x + c$

$\because f'(0) = 1 - 0 + c = 0, \quad c = -1$

$\therefore f'(x) = e^{2x} - 2x - 1$

(iii) $f(x) = \frac{e^{2x}}{2} - x^2 - x + c$

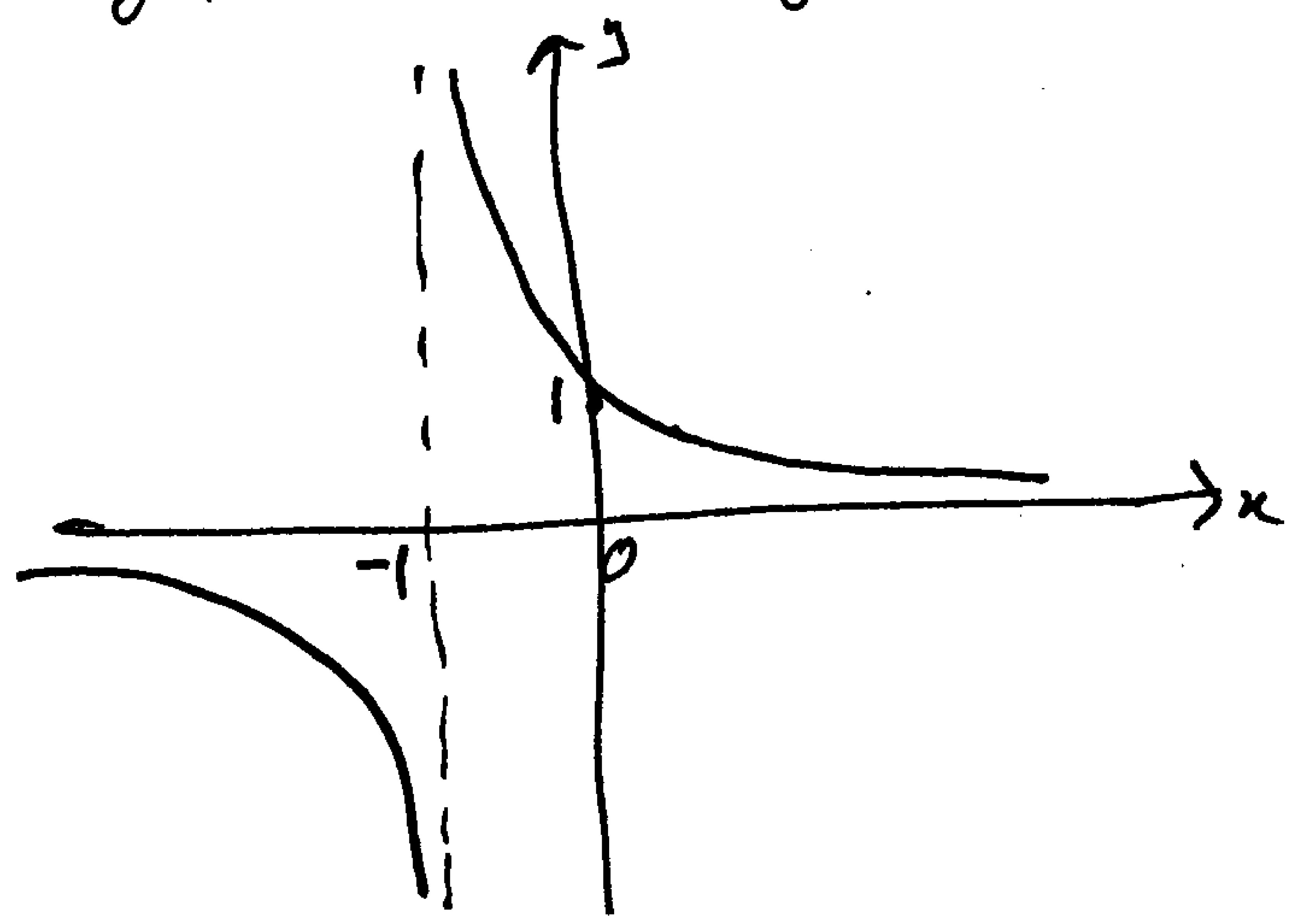
$\because f(0) = \frac{1}{2} - 0 + c = 0, \quad c = -\frac{1}{2}$

$\therefore f(x) = \frac{e^{2x}}{2} - x^2 - x - \frac{1}{2}$

$\therefore f(1) = \frac{e^2}{2} - 1 - 1 - \frac{1}{2} = \frac{e^2-5}{2}$

Question 5

(a) (i) $x \neq -1,\ y \neq 0$; $x = 0, y = 1$



(ii) $V = \pi \int_0^k \left(\frac{1}{x+1}\right)^2 dx = \pi \int_0^k (x+1)^{-2} dx$

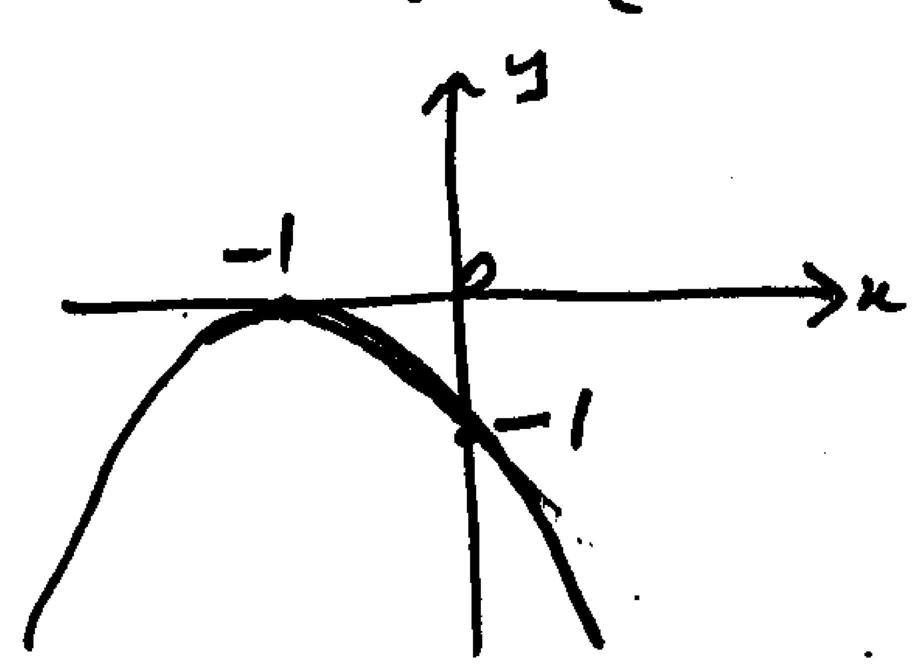
$= \pi \left[\frac{(x+1)^{-1}}{-1}\right]_0^k$

$= \pi \left[-\frac{1}{x+1}\right]_0^k = \pi\left(-\frac{1}{k+1} + 1\right)$

$= \pi\left(1 - \frac{1}{k+1}\right) < \pi$ since $\frac{1}{k+1} > 0$

(b) (i) $y = -x^2 - 2x - 1 = -(x^2 + 2x + 1) = -(x+1)^2$

$\therefore$



(ii) $\Delta = (k+2)^2 - 4(-1)(-1) = (k+2)^2 - 4$

$= k^2 + 4k$

$= k(k+4) \geq 0$ for real roots

$\therefore k \leq -4$ or $k \geq 0$

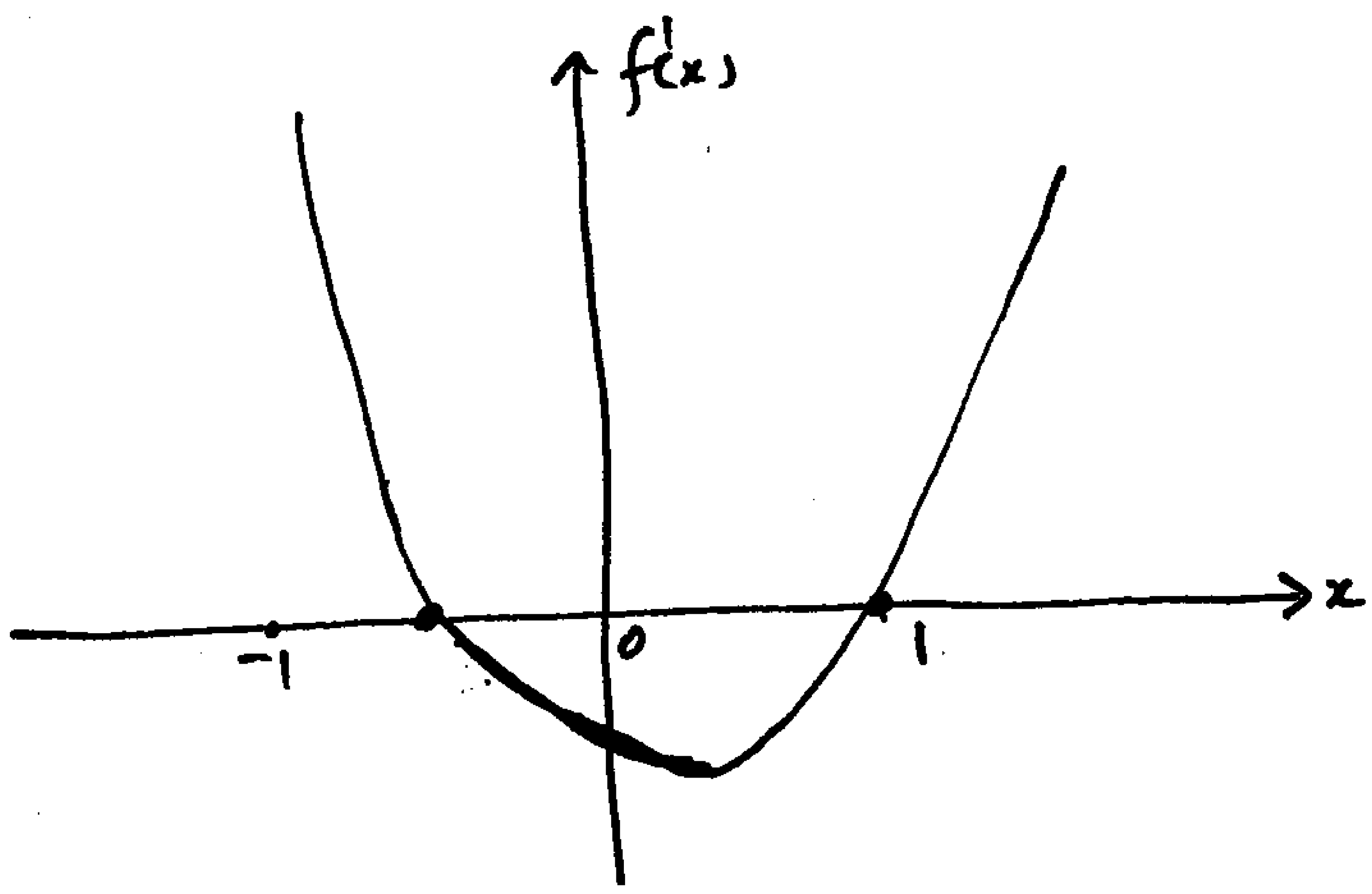
Question 6

(a) $\frac{dy}{dx} = (2x+2)\cos(x^2+2x)$

For $x = -2$, $y = \sin 0 = 0$, $\frac{dy}{dx} = -2\cos 0 = -2$

$\therefore$ tangent is $y = -2(x+2)$

(b)

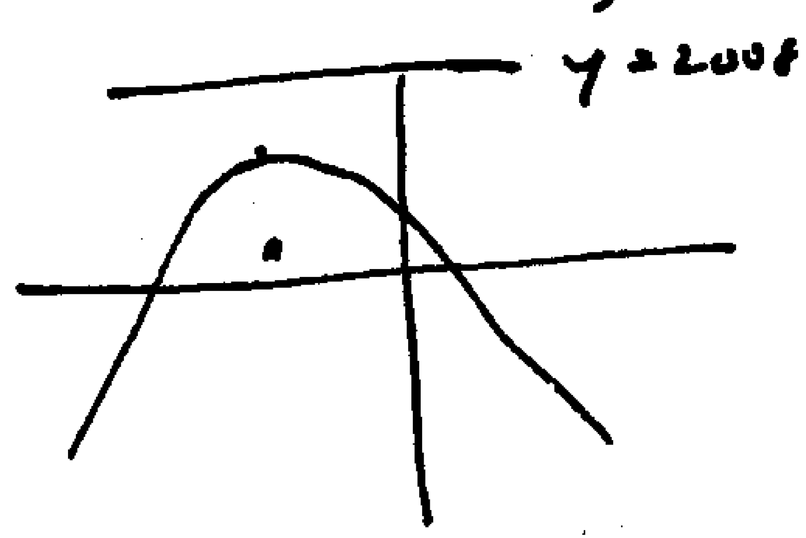


(c) P to the focus = P to the directrix

$= 2008 - 998 = 1010$

OR (sadly) $V = (-2, 1008)$; $a = 1000 \Rightarrow S = (-2, 8)$

$\Rightarrow$



$\therefore PS = \sqrt{200^2 + 990^2} = 1010$

(d) $\frac{a(b-c) - b(a-c) + c(a-b)}{(a-b)(a-c)(b-c)} = \frac{ab - ac - ab + bc + ac - bc}{(a-b)(a-c)(b-c)} = 0$

(e) Since $-1 \le \sin x \le 1$ and $\ln x$ is an increasing function,

range y : $\ln(2-1) \le y \le \ln(2+1)$

i.e. $0 \le y \le \ln 3$

Question 7

(a) $\int_0^{\frac{\pi}{3}} 2\tan^3 x \, dx \approx \frac{1}{6} \cdot \frac{\pi}{3} \left(0 + 2\tan^3 \frac{\pi}{3} + 8 \tan^3 \frac{\pi}{6} \right)$

$= 2.08$, 2 d.p.

(b) (i) $\frac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$

$= 2\tan x (1 + \tan^2 x) = 2\tan x + 2\tan^3 x$

(ii) $\frac{d}{dx}(\sec^2 x) = \frac{d}{dx}(1 + \tan^2 x) = 2\tan x + 2\tan^3 x$ or equivalent

(iii) $2\tan^3 x = \frac{d(\tan^2 x)}{dx} - 2\tan x$

$\therefore \int_0^{\frac{\pi}{3}} 2\tan^3 x \, dx = \left[\tan^2 x\right]_0^{\frac{\pi}{3}} - 2\int_0^{\frac{\pi}{3}} \tan x \, dx$

$= (\sqrt{3})^2 - 0 + 2 \int_0^{\pi/3} \frac{-\sin x}{\cos x} \, dx$

$= 3 + 2\left[\ln \cos x\right]_0^{\pi/3}$

$= 3 + 2\left(\ln \frac{1}{2} - \ln 1\right)$

$= 3 + 2\ln 2^{-1} = 3 - 2\ln 2$

(c) (i) S_∞ exists for $-1 < 2x < 1$ i.e. $-\frac{1}{2} < x < \frac{1}{2}$

(ii) Put $S_\infty = \frac{1}{1-2x} = 12$

$\therefore 1 - 2x = \frac{1}{12}$

$2x = \frac{11}{12}$

$x = \frac{11}{24} < \frac{1}{2}$

$\therefore$ from (i), it's possible

Question 8

(a) $A = 2\int_0^{16} x \, dy$, $x \geq 0$: $x^4 = \frac{y}{16}$

ie. $x = \frac{y^{\frac{1}{4}}}{2}$

$\therefore A = \int_0^{16} y^{\frac{1}{4}} dy = \frac{4}{5}\left[y^{\frac{5}{4}}\right]_0^{16} = \frac{4}{5} \cdot 2^5 = \frac{128}{5}$

(b) (i) Albert made $\$ 2000(.12)(20) + 2000(.12)(19) + \cdots + 2000(0.12)$

$= \$240(1+2+\cdots+19+20)$

$= \$ 240 \cdot \frac{20}{2} \cdot 21 = \$50\,400$ interest

(ii) Betty's total over the 20 years

$= \$2000(1.07)^{20} + 2000(1.07)^{19} + \cdots + 2000(1.07)^2 + 2000(1.07)$

$= \$2000(1.07)[1 + 1.07 + 1.07^2 + \cdots + 1.07^{19})$

$= \$ 2000(1.07)\frac{1.07^{20}-1}{.07} = \$87730$, nearest dollar

$\therefore$ Interest made $= \$87730 - \$2000 \times 20 = \$47730$

$\therefore$ Albert's decision was better

(c) $\therefore 2\ln x - \ln x - 12 = 0$

$\ln x = 12$

$x = e^{12}$

Question 9

(a) (i) $S = 2\pi r h + \pi r^2$: $\pi r^2 h = 1728\pi$

$$h = \frac{1728}{r^2}$$

$$\therefore S = 2\pi r \cdot \frac{1728}{r^2} + \pi r^2 = \pi\left(r^2 + \frac{3456}{r}\right)$$

(ii) $S = \pi(r^2 + 3456 r^{-1})$

$$\frac{dS}{dr} = \pi(2r - 3456 r^{-2})$$

$$\frac{d^2S}{dr^2} = \pi(2 + 6912 r^{-3})$$

$\frac{dS}{dr} = 0 \Rightarrow r - \frac{1728}{r^2} = 0$ or $r^3 = 1728$

$\Rightarrow r = \sqrt[3]{1728} = 12$

For $r = 12$, $\frac{d^2S}{dr^2} > 0$ (indeed $\frac{d^2S}{dr^2} > 0$ for all $r > 0$)

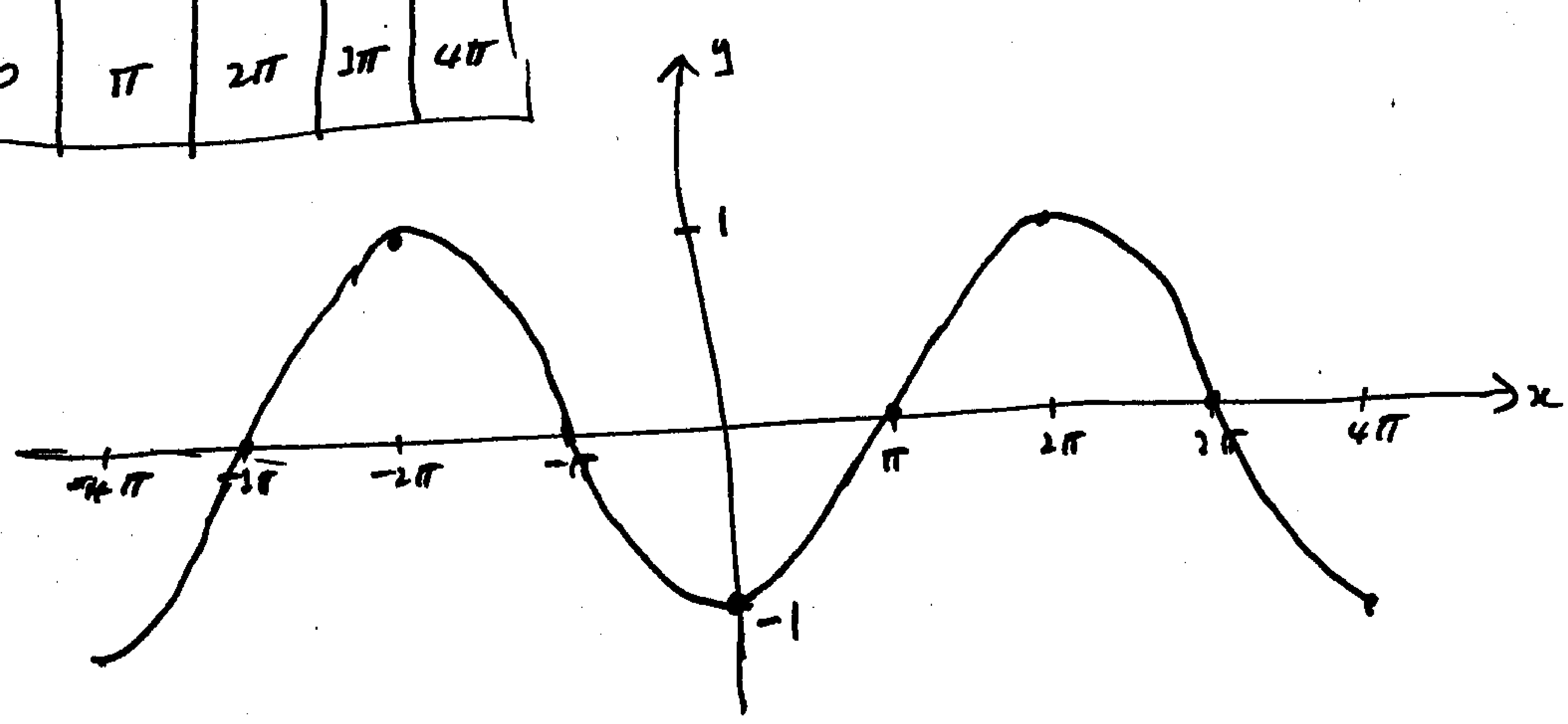
$\Rightarrow$ minimum S when $r = 12$

$\therefore$ min $S = \pi\left(12^2 + \frac{3456}{12}\right)\text{cm}^2 = 432\pi\ \text{cm}^2$

(b)

$\frac{x}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	-1	0	1	0	-1
x	0	π	2π	3π	4π

$\therefore$



(c) (i) $-5 \le x+2 \le 5$

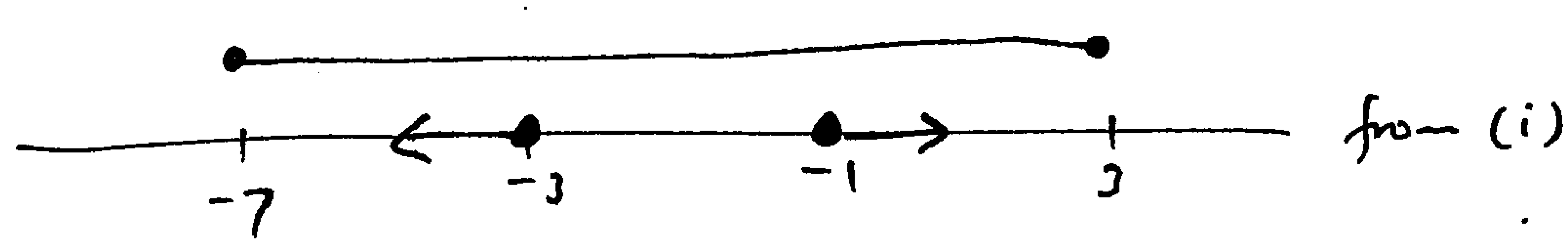
$\therefore \; -7 \le x \le 3$

(ii) For $|x+2| \geq 1$

we have $-1 \geq x+2 \geq 1$

$-3 \geq x \geq -1$ ie. $x \le -3$ or $x \geq -1$

$\Rightarrow$



$\therefore \; -7 \le x \le -3$ or $-1 \le x \le 3$

<u>OR</u> $1 \le |x+2| \le 5$

$\Rightarrow \; 1 \le x+2 \le 5$ or $-5 \le x+2 \le -1$

$\therefore \; -1 \le x \le 3$ or $-7 \le x \le -3$

Question 10

(a) (i) $\therefore \frac{1}{2}.AB.h = \frac{1}{2}.AB.a + \frac{1}{2}.BC.b + \frac{1}{2}.AC.c$,

where $AB = BC = AC$, Δ equilateral

$\therefore h = a + b + c$

(ii) OB is $y - 0 = \tan 60°(x - 0)$

i.e. $y = \sqrt{3}x$

(iii) $\perp$ distance from $P(\sqrt{3},1)$ to $\sqrt{3}x - y = 0$ is

$$\frac{\sqrt{3}.\sqrt{3} - 1}{\sqrt{\sqrt{3}^2 + 1^2}} = \frac{3-1}{\sqrt{4}} = 1$$

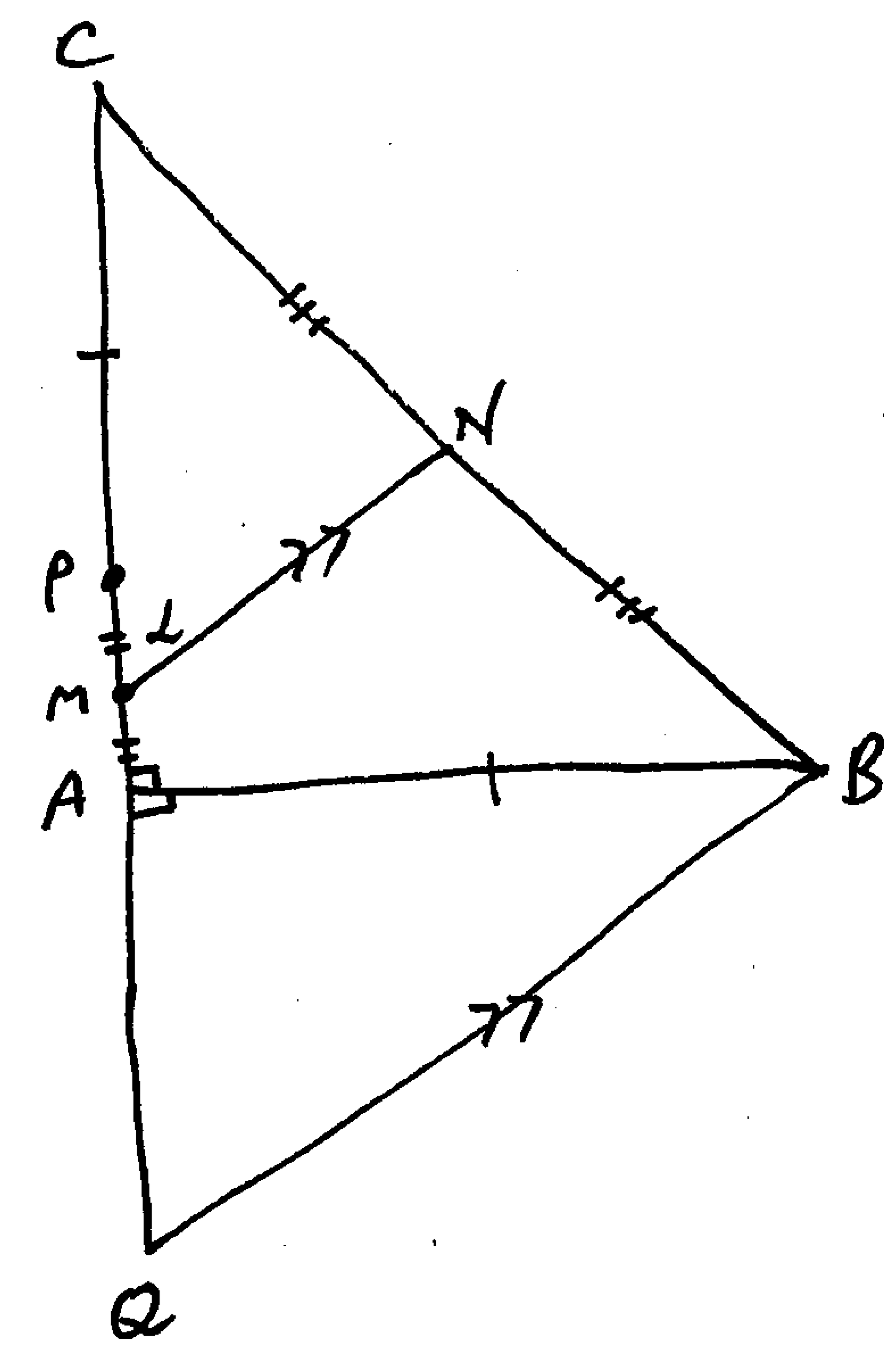
(iv) Using notation in (i), $c = 1$ from (iii)

$a = 1$ since $P = (\sqrt{3}, 1)$

at B, $x = 2$ $\therefore y = 2\sqrt{3} = h$

$\therefore$ distance from P to $AB = 2\sqrt{3} - 1 - 1 = 2\sqrt{3} - 2$

(b)



$\angle BQA = \angle NMC = \alpha$, corresponding $\angle$s in $\parallel$ lines

$\frac{BN}{NC} = \frac{QM}{MC} = 1:1$, ratio intercept theorem in $\parallel$ lines

$\therefore QM = MC$

But $AM = MP$, data

$\therefore QA = PC = AB$, data

$\therefore \Delta ABC$ is right-angled and isosceles.

$\Rightarrow \angle Q = \angle B = 45°$, base angles equal

$\therefore \alpha$ is the fixed value $45°$